May 2, 2026

Intermediate test - Signal processing and information theory

Question A

1. What is the difference between an energy signal and a power signal?
2. Can an energy signal be a member of an ergodic process? Give reasons for your answer.

Question B A complex number $\underline{a} = 2 - j$ is given

1. Calculate $|\underline{a}|$
2. Draw $\underline{b} = \underline{a} \exp\left(j\frac{\pi}{2}\right)$ in the complex plane.
3. Verify answer 2 by applying Euler's formula, or alternatively if already applied.

Question C Visit the Falstad website and request a sawtooth display

1. Read and report the value of the fundamental frequency F and the fourth harmonic;
2. Read and report the respective amplitudes and phases;
3. Express the value of the corresponding Fourier coefficients, in magnitude and phase.

Question D What type of trigonometric series do you get for an odd real signal?

Question E Why does the quadratic norm of an energy (or power) signal-vector equal its total energy (or power)?

Question F What does it mean when we say that the Fourier transform is a unitary operator?

Question G Let us consider the signal $x(t) = \cos\left(2\pi\frac{t}{T}\right)$ placed at the input to a filter with impulse response $h(t) = rect_{\frac{3}{2}T}(t)$

1. Is this a causal filter?
2. Draw (one above to the other and in the same scale) $x(t)$ and $h(t)$;
3. Express the transforms $X(f)$ and $H(f)$;
4. Evaluate the transform $Y(f)$ of the filter output using the frequency product, performing the intermediate steps and commenting on the reasoning;
5. Get the $y(t)$ output;
6. How does $y(t)$ changes when you replace $h(t)$ with another filter $g(t)$ that is equivalent to the first, but causal?

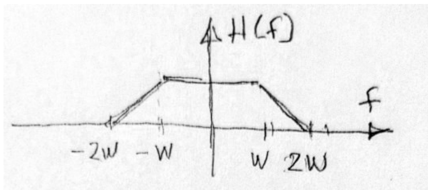
Question H Let us address the concept of time windowing, where a signal $x(t)$ is multiplied by another signal $w(t)$, of limited duration, obtaining a third signal $y(t) = x(t) \cdot w(t)$:

1. What is the relationship between $X(f)$ and $Y(f)$?
2. What is meant by spectral leakage? What criteria must $W(f)$ satisfy for it to be reduced?
3. Windows that offer low spectral leakage often suffer from low spectral resolution power. Define what is meant by spectral resolution.

4. After visiting the Wikipedia page on windows, examine the spectrum graph (in dB) of the Hamming, Blackman, and Nuttall windows, and verify the previous statement, drawing up a double ranking for the three, according to the criterion of minimum infiltration, and maximum resolution.

Question I Consider a band-limited signal $x(t)$, whose spectrum can be described as $X(f) = tri_{2W}(f)$, and we wish to subject it to a sampling process.

1. Assuming that we have a restitution filter $H(f)$ with the trapezoidal frequency response shown in the figure, what is the minimum sampling frequency that should be used to avoid aliasing?
2. draw the corresponding $X^{\bullet}(f)$



Let us now consider the presence of a noise process $n(t)$, with a smaller amplitude and bandwidth $\gg W$, superimposed on the signal $x(t)$.

3. having an anti-aliasing filter with the same $H(f)$ as the restitution filter, what is the new minimum sampling frequency that can be adopted? Draw the new $X^{\bullet}(f)$.

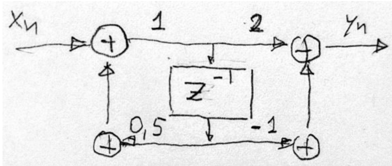
Question L What are the differences between the DTFT and DFT of a sampled signal, in terms of...

- discrete or continuous nature of the frequency representation
- periodicity or not of the inverse transform
- numerical computability
- approximation (or not) of the spectrum of a power signal
- relation to the Zeta transform

Question M Go to the falstad/dfilter website: by default, we find (read at the bottom) a 120^{th} order low-pass FIR filter, with a cutoff frequency of 2205 Hz, which uses a Hamming window.

1. In what aspect of the filter synthesis does this window play a role?
2. Describe the advantages and disadvantages of an FIR filter compared to an IIR filter
3. identify the attenuation in dB corresponding to the first side lobe, and the frequency at which the lobe is located *(hover your mouse over the frequency response, and read below)*
4. repeat the operation by changing the window selection, to determine
   (a) the window that offers the worst attenuation in the stop band
   (b) the best window, according to the same criteria *(it may be necessary to increase the scale in dB)*
5. What is the downside to choosing a window with high stop band attenuation? How can this be remedied? *(Experiment with the applet.)*

Question N Let us take the canonical architecture of a numerical filter shown in the figure.



1. Express the $H(z)$ associated with it
2. Express the corresponding finite difference equation
3. Is this a stable filter? Why?

Question O Consider a white ergodic Gaussian process x(t), band-limited between $-W$ and W, and two random variables x_1 and x_2 extracted from it at two time instants separated by an interval T. The corresponding bivariate probability density is described by the mean vector $m_x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and the covariance matrix $\Sigma_x = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.

1. Are the random variables x_1 and x_2 statistically independent? Why?
2. What is the power $\mathcal{P}_x$ of the members of the random process?
3. Determine the expression of the power density spectrum $\mathcal{P}_x(f)$
4. Determine the expression of the autocorrelation function $\mathcal{R}_x(f)$
5. Find the minimum value of the interval T so that it results $\sigma_{x_1 x_2} = 0$

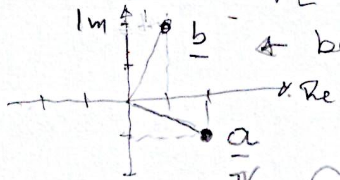
| Official answers to the SP&IT test 2 May 2026 |

(A) 1) $x(t)$ is an energy signal if $\mathcal{E}_x = \int_{-\infty}^{\infty} s(t)s^*(t)\,dt < \infty$

$x(t)$ is a power signal if $P_y = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$

2) No, it can't. Because an ergodic process needs its members to be stationary, meaning that they must preserve the statistical description (mean, variance ...) of values extracted from them independent from the time instant when they are extracted

(B) 1) $|\underline{a}| = \sqrt{\underline{a}\cdot\underline{a}^*} = \sqrt{[Re\{\underline{a}\}]^2 + [Im(\underline{a})]^2} = \sqrt{4+1} = \sqrt{5}$

2) 

because $e^{J\frac{\pi}{2}} = J$, so $\underline{b} = \underline{a}\cdot J = 2J+1$

3) in fact $e^{J\pi/2} = \cos\frac{\pi}{2} + J\sin\frac{\pi}{2} = J$; alternatively, $e^{J\pi/2}$ adds a $\frac{\pi}{2}$ phase to $\underline{a}$ (the phase of a sum of complex numbers is the sum of the phases

(C) 1) $F = 221.72$ Hz $\quad f_4 = 4F = 886.9$ Hz

2) F amplitude $= 0.6366$ $\quad F$ phase $= \emptyset$

f_4 amplitude $= 0.1591$ $\quad f$ phase $= \pi \Rightarrow$ which gives the minus sign

the applet represents the periodic signal as a sum of sines, and this is correct, because the signal is odd

3) We must click on the check box Mag/Phase View by which we observe all positive magnitudes and alternating phases. In fact the formula

$x(t) = X_0 + 2\sum_{n=1}^{\infty} M_n \cos(2\pi n F t + \varphi_n)$ is applied,

so that $|X_1| = 0.6366 \quad \varphi_1 = -\frac{\pi}{2} = 1.5677$

$|X_4| = 0.1591 \quad \varphi_4 = \frac{\pi}{2}$

(D) A real signal $x(t)$ has a Fourier Amplitude Density with conjugate simmetry, i.e. $X(f) = X^*(-f)$ or $|X(f)| = |X(-f)|$ and $\varphi_x(f) = -\varphi_x(-f)$

If it is real odd, it's spectrum is completely imaginary, so that we can write $X(f) = -X(-f)$. If $x(t)$ is periodic as in the previous question, then it can be written in terms of only sines with zero phase, or with only cosines, with alternating $\pm\frac{\pi}{2}$ phases in contiguous harmonics

(E) • The quadratic norm is a vector space concept when a norm operator is defined for it, so that it can be evaluated as $\|\underline{x}\|^2 = \langle \underline{x}, \underline{x} \rangle$

• The energy (or power) of a signal is defined as given in the answer to question (A)

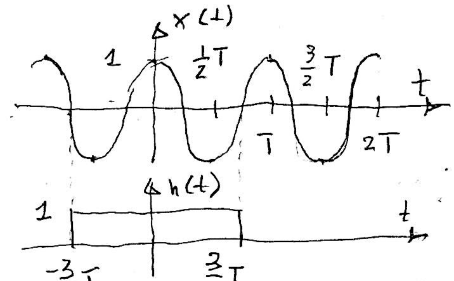
• When we cast signals into vectors of a signal (vector) space, we define for them a dot product as $\langle \underline{x}, \underline{y} \rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$ for energy signals or $\langle \underline{x}, \underline{y} \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$ for power ones.

• When we evaluate the dot product of a signal by itself, this is the definition of $\|\underline{x}\|^2$, and due to the above definitions for the dot product in the signal space, we recognize that we are actually computing their energy (power)

(F) It's a consequence of the Parseval theorem, which states the equality $\int x(t)x^*(t)\,dt = \int X(f)X^*(f)\,df$, which is the equivalence of the squared norm in both the time and frequency domains. In this sense, the Fourier Transform operator doesn't change the norm of the same signal vector after the change of basis

(G) 1) No, it isn't, because $h(t) \neq 0$ with $t < 0$

2)



3) by knowing that $x(t) = \dfrac{e^{j2\pi\frac{1}{T}t} + e^{-j2\pi\frac{1}{T}t}}{2}$

we obtain $X(f) = \frac{1}{2}\left[\delta(f - \frac{1}{T}) + \delta(f + \frac{1}{T})\right]$; by using known results we get. $H(f) = \frac{3}{2}T \operatorname{sinc}\left(\frac{3}{2}T \cdot f\right)$

4) $Y(f) = X(f)\,H(f) = \frac{1}{2}\left[\delta(f - \frac{1}{T}) + \delta(f + \frac{1}{T})\right] \cdot \frac{3}{2} T \operatorname{sinc}\left(\frac{3}{2}T \cdot f\right) = \left(\begin{matrix}\text{sinc is} \\ \text{even}\end{matrix}\right)$

$= \cancel{2} \cdot \frac{3}{2} T \operatorname{sinc}\left(\frac{3}{2}\right) \cdot \frac{1}{\cancel{2}}\left[\delta(f - \frac{1}{T}) + \delta(f + \frac{1}{T})\right]$

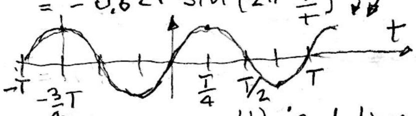
now remember that $\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$ so that

$\operatorname{sinc}\left(\frac{3}{2}\right) = \dfrac{\sin(\frac{3}{2}\pi)}{\frac{3}{2}\pi} \cong \dfrac{-1}{1{,}5 \cdot 3{,}14} = -0{,}21$ so that $\frac{3}{2}T(-0{,}21) = -0{,}31\,T$

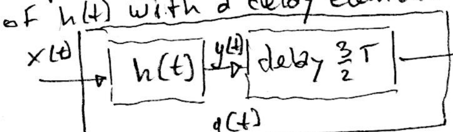
and $Y(f) = -0{,}31 \cdot T \cdot \left[\delta(f - \frac{1}{T}) + \delta(f + \frac{1}{T})\right]$

5) $y(t) = \mathcal{F}^{-1}\{Y(f)\} = -0{,}62\,T\ \dfrac{e^{j2\pi\frac{1}{T}t} + e^{-j2\pi\frac{1}{T}t}}{2} = -0{,}62\,T \cos\left(2\pi \frac{t}{T}\right)$

6) for $h(t)$ to be causal, it has to be substuted by $g(t) = h(t - \frac{3}{4}T) = rect_{\frac{3}{2}T}(t - \frac{3}{4}T)$, that as well known, introduces a linear phase term $e^{-j2\pi\frac{3}{4}Tf}$ to $H(f)$, and to $Y(f)$ as well, to which it corresponds a new $y'(t) = -0{,}62T\cos\left[2\pi(t - \frac{3}{4}T)\frac{1}{T}\right] =$
$= -0{,}62T\sin(2\pi\frac{t}{T})$ ∂∂



This can be seen also if we express $g(t)$ as the cascade of $h(t)$ with a delay element



so that the new $y(t)$ is delayed by the same amount

H) 1) A multiplication in time domain corresponds to a convolution in frequency, so that $Y(f) = X(f) * W(f)$ where $W(f) = \mathcal{F}\{w(t)\}$

2) If $W(f)$ is not an impulse, every frequency content in $X(f)$ will be smeared by "tails" of $W(f)$, because of the frequency convolution. For istance, is $x(t)$ is a single tone then $X(f)$ is made by two pulses, and $Y(f)$ is made by $W(f)$ centered at the tone frequency. So, we can say that spectral leakage is the appearance of frequency content that was not present in the unwindowed signal.

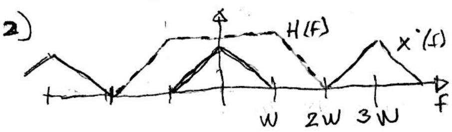
• In order to reduce spectral leakage $W(f)$ needs to have tails that decrease fast in frequency

3) Spectral resolution is the ability of resolve two close frequencies, i.e. to be able of distinguish them one from the other, also if the original signal has been windowed. This ability depends on the width of the main lobe around the origin of $W(f)$.

4)

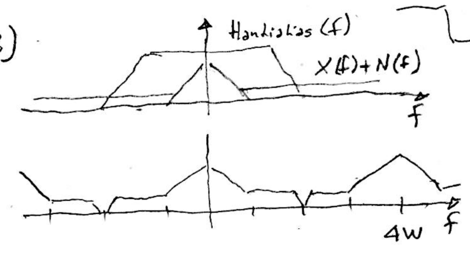
Window	Leakage	Resolution
Hamming	bad	good
Blackman	fair	fair
Nuttal	excellent	bad

I) 2)



1) In this case we need a 3W minimum sampling frequency, because in this way the spectral replica is outside of the $H(f)$ bandwidth

3)



In this other case we need a 4W minimum sampling freq, because the antialias filter cuts noise at $f > 2W$, so now the bandwidth of the signal $x(t)+n(t)$ to be sampled is double

(L)

FEATURE	DTFT	DFT
nature of the freq. representation	continous	Discrete
periodicity of the inverse transform	No	yes
numerical computability	No	yes
approximation of the true spectrum	true spectrum	approximated
relation with the z transform	$X(z)\Big\|_{z=e^{jw}} = X^*(f)\Big\|_{f=\frac{w}{2\pi T_c}}$	$X_m \simeq X(e^{jw})\Big\|_{w=2\pi\frac{m}{N}}$

(M) 1) A window is used during the design of a FIR filter, starting from the samples of the h(t) of an analog filter we want to emulate. Instead of using the samples as such, we multiply them by the samples of a (non-rectangular) window, i.e. $h_n = h(nT_c)\cdot w_n$. This allows to obtain a better attenuation in the stop band, at the expenses of slightly larger transition region

2) FIR filters can easily have a simmetric impulse response, to which it corresponds a filter with a linear phase, whereas this feature is hard to obtain through IIR filters.
IIR filters have a lower computational complexity

3) attenuation = $-52{,}07$ dB at 2940 Hz

4) (a) - the worst attenuation = -22.53^{dB} @ 2594 Hz for a rectangular window
(b) - the best " $= -131{,}85$ dB @ 3934 Hz
is attained by a Kaiser window

5) the down side is to have a slight larger transition region

(N) 1) $H(z) = \frac{2 - z^{-1}}{1 - 0{,}5\, z^{-1}}$ 2) $y_n = 0{,}5\, y_{n-1} + 2\, x_n - x_{n-1}$

3) Yes, it is a stable filter, because its pole at $z = 0{,}5$ lies inside of the unit circle

(O) 1) They are uncorrelated because $\sigma_{x_1x_2} = 0$ as given by the Σ_x; but their joint pdf is gaussian, so they are also statistically independent

2) As the process is ergodic, the power of its members is equal to the correspondent ensamble average, which in this case is $m_x^{(2)} = \sigma_x^2 + (m_x)^2$. We have $\sigma_x^2 = 4$ as given from Σ_x, and $m_x = 2$ as given by the mean vector, so
$P_x = m_x^{(2)} = \sigma_x^2 + (m_x)^2 = 4 + 4 = 8$

3) $P_x(f) = \underbrace{\frac{\sigma_x^2}{2W}\, \mathrm{rect}_{2W}(f)}_{\text{zero-mean component}} + \underbrace{(m_x)^2\, \delta(f)}_{\text{Power of the continous component}} = \frac{4}{2W}\, \mathrm{rect}_{2W}(f) + 4\, \delta(f)$

4) $R_x(\tau) \underset{\text{wiener}}{=} \mathcal{F}^{-1}\{P_x(f)\} = \frac{4}{2W}\, 2W\, \mathrm{sinc}(2Wt) + 4 = 4\,\mathrm{sinc}(2Wt) + 4$ (4)

5) It is at the first zero of the sinc, i.e. $T_{min} = \frac{1}{2W}$ [$\sigma_{x_1x_2}$ is a central moment]