June 3, 2022

Third intermediate test - Signal processing and information theory

Question A Write the expresion for the mean m_x and the variance σ_x^2 of a r.v. x in terms of its moments with respect to the p.d.f. $p_X(x)$.

Question B Do you know how to *saturate* the joint p.d.f. $p_{\boldsymbol{X}}(\boldsymbol{x})$ of a multivariate r.v. $\boldsymbol{x} = [x_1, x_2, x_3]$ in order to obtain the p.d.f. of one (or more) of its constituent marginal random variables? (use continous or discrete domain notation, as you prefer)

Question C

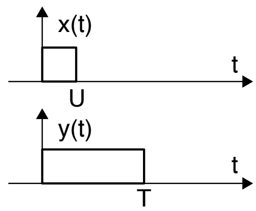
1. Write the equivalence between a time average evaluated on the j^{th} member $x_j(t)$ of a stationary process, and the expected value $E\{x_j\}$ for a r.v. x_j obtained by sampling it randomly;
2. explain why (if the process is ergodic) the same is true for any member.

Question D

1. Describe the difference between correlation and covariance for a pair of r.v. x, y;
2. why does it happens that two statistically independent r.v. have zero covariance?
3. What is the only case in which uncorrelation implies statistical independence?
4. Can you tell in words why (3) holds true?

Question E Draw the result of the intercorrelation between $x(t)$ and $y(t)$, i.e.

$$\mathcal{R}_{xy}(\tau) = \int x(t)\, y(t+\tau)\, dt$$



Question F Given a zero-mean, band-limited and ergodic process with $\mathcal{P}_x(f) = \frac{N_0}{2} rect_{2W}(f)$

1. what is the autocorrelation function $\mathcal{R}_x(\tau)$ for any of the process members? (probably it is better if you draw it)
2. Show that $\mathcal{R}_x(0)$ is equal to the power $\mathcal{P}_x$ of any of the process members.
3. Let us now sample a member $x(t)$ of the process with period $T = 1/4W$: write the expression of the correlation between the first and the second sample, and between the first and the second, that is of $\mathcal{R}_x(0)$, $\mathcal{R}_x(T)$ and $\mathcal{R}_x(2T)$.
4. Now let the process to be gaussian, let's call these consecutive samples x_1, x_2 and x_3, and consider them as the marginals of a tri-variate gaussian r.v. $\boldsymbol{x} = [x_1, x_2, x_3]$. Then write the covariance matrix Σ with elements $\sigma_{ij} = E\{x_i x_j\}$ (as they are zero-mean).
5. which marginal x_i can be said statistically independent of which x_j?

Question G Let us define the function of a biochemical reaction as the observation of a r.v. y whose value depends on the value of a r.v. x as a cause (can be concentrations). Let us restrict our attention to a couple of binary r.v., i.e. such that $x \in \{x_{0,}, x_1\}$ and $y \in \{y_{0,}, y_1\}$. Finally, let us consider the realizations of x without memory, and described by an *a priori* p.d.f. $p(x_0)$ and $p(x_1)$, and the dependence described by a set of *forward* conditional probabilities $\begin{bmatrix} p(y_0/x_0) & p(y_0/x_1) \\ p(y_1/x_0) & p(y_1/x_1) \end{bmatrix}$.

1. Define the entropy $H(X)$ of the sequence x;
2. define the mutual information between (for instance) x_0 and y_1;
3. define the *average* mutual information $I(X;Y)$ between elements of x and y;
4. why the difference $H(X) - I(X;Y) = H(X/Y)$ (the average uncertainity about $x$when y is known) is called *equivocation*, and when does it become zero?

Question H

Intermediate Test of Signal Processing and Information Theory

03-06-2022

1) Given an undirected graph $\mathcal{G}$ composed of N nodes and with associated Laplacian matrix $\mathbf{L}$:

   a) Define the Graph Fourier Transform (GFT) and its Inverse (IGFT) of a graph signal $\boldsymbol{s}: \mathcal{V} \to \mathbb{R}^N$ observed over the nodes of $\mathcal{G}$.

   b) When the graph signal $\boldsymbol{s}$ is said band-limited over $\mathcal{G}$ with bandwidth K?

1 Soluzioni alla terza prova intermedia SP&IT – 3 giugno 2022

(A)

$$m_x = E\{x\} = \int x\, p_x(x)\, dx$$

$$\sigma_x^2 = E\{(x-m_x)^2\} = \int (x-m_x)^2 p_x(x)\, dx$$
$$= m_x^{(2)} - (m_x)^2$$

(B)

continuous $\quad p(x_i) = \int_{x_J}\int_{x_K} p(x_i x_J x_K)\, dx_J\, dx_K$

discrete $\quad p(x_i) = \sum_{J=1}^{n_J}\sum_{K=1}^{n_K} p(x_i x_J x_K)$

(C) for example, the first order time average (mean) is equal
1) to the mean of the r.v., i.e.

$$\overline{x_J(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_J(t)\, dt = E\{x\} = \int x_J\, p_x(x_J)\, dx$$

2) because for an ergodic process we have $p_x(x_J) = p(x)$ independent of J, i.e. the p.d.f. is the same for all members

(D) Correlation is the expected value of the product, or
1) the mixed moment of order (1,1)

$$\mathrm{Corr}(x,y) = m_{xy}^{(1,1)} = E\{x \cdot y\} = \int x \cdot y \cdot p_{xy}(x,y) \cdot dx\, dy$$

Covariance is the central mixed moment of the same order, i.e.

$$\sigma_{xy} = \mu_{xy}^{(1,1)} = E\{(x-m_x)(y-m_y)\} = \int (x-m_x)(y-m_y)\, p(xy)\, dx\, dy$$

2) If x and y are statistically independent then

$p_{xy}(x,y) = p(x)p(y)$ and thus $m_{xy}^{(1,1)} = \int x p(x) dx \cdot \int y p(y) dy = m_x \cdot m_y$

It follows that

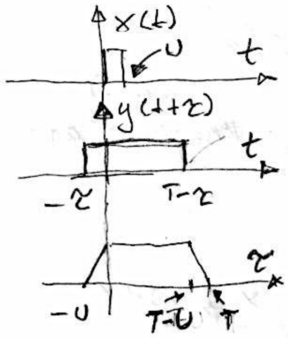
$$\sigma_{xy} = E\{xy\} - E\{x m_y\} - E\{m_x y\} + E\{m_x m_y\} =$$
$$= m_{xy}^{(1,1)} - m_x m_y - m_x m_y + m_x m_y = m_{xy}^{(1,1)} - m_x m_y =$$
$$= \emptyset$$

(1)

3) uncorrelation implies statistical independence only if the r.v. are jointly gaussian

4) It is because uncorrelated marginal r.v. of a multivariate gaussian p.d.f. makes the covariance matrix Σ to be diagonal, hence the joint p.d.f. is factored on the product of the p.d.f. of the marginal p.d.f., which exactly is the definition of statistical independence

(E)



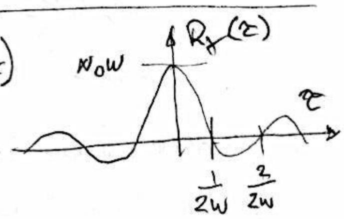
$y(t+\tau)$ è un anticipo e la y è traslata a sinistra. L'area del prodotto però è quella per τ positivo, quindi si disegna a destra.

- Quando $\tau < -U$ allora $y(t+\tau)$ sta a destra di $x(t)$
- Quando $\tau = T$ allora $y(t+\tau)$ è tutto a sinistra di $x(t)$

(F)

Teo di Wiener

1) $R_x(\tau) = \mathcal{F}^{-1}\{P_x(f)\} = \frac{N_0}{2} 2W \operatorname{sinc}(2W\tau)$



2) By definition

$R_x(0) = \lim_{T\to\infty} \frac{1}{T}\int x^2(t)\,dt = P_x$ but also

by the initial value property of the fourier transform,

in fact $R_x(0) = \frac{N_0}{2} 2W \operatorname{sinc}(2W \cdot \emptyset) = N_0 W$

$P_x = \int P_x(f)\,df = \frac{N_0}{2}\int_{-W}^{W} df = \frac{N_0}{2} 2W = N_0 W$

3) $R_x(0) = N_0 W$

$R_x(T_c) = R_x\left(\frac{1}{4W}\right) = N_0 W \operatorname{sinc}\left(\frac{1}{2}\right) = N_0 W \beta$

$R_x(2T_c) = R_x\left(\frac{1}{2W}\right) = \emptyset$

4) $$\Sigma = N_0 W \begin{bmatrix} 1 & \beta & \emptyset \\ \beta & 1 & \beta \\ \emptyset & \beta & 1 \end{bmatrix}$$

5) For a gaussian process incorrelation implies stat. indep., therefore

x_1 and x_3 are stat. indep.

x_1 and x_2 are Not st. ind.

x_2 and x_3 " " " "

6) 1) $H(X) = E\{I(x)\} = \sum p(x_i) \log_2 \frac{1}{p(x_i)} = -\sum p(x_i) \log_2 p(x_i)$

2) $I(x_0; y_1) = \log_2 \frac{p(x_0, y_1)}{p(x_0)p(y_1)} = \log_2 \frac{p(y_1/x_0)}{p(y_1)}$

3) $I(X;Y) = E\{I(x,y)\} = \sum_{i=0}^{1} \sum_{J=0}^{1} \underbrace{p(x_i, y_J)}_{p(y_J/x_i)p(x_i)} \log_2 \underbrace{\frac{p(x_i, y_J)}{p(x_i)p(y_J)}}_{\frac{p(y_J/x_i)}{p(y_J)}}$

4) Se avessimo utilizzato dei numeri (rendendo l'esercizio più difficile) avremmo verificato che

$H(X) - I(X;Y) = H(X/Y) \geq 0$, ovvero che l'informazione mutua media è sempre inferiore all'entropia di sorgente a causa della perdita di informazione causata dalla natura non deterministica del legame tra X ed Y, dovuto al "rumore" associato ai molteplici fattori che influenzano la reazione chimica. Pertanto tale differenza $H(X/Y)$ è chiamata "equivocazione" intendendo la fonte di equivoco dovuto al rumore, e si annulla nel caso di una dipendenza deterministica tra X ed Y, ovvero di una matrice di probabilità condizionate diagonale.

3

Question H

Solution:

a) Given the symmetric Laplacian matrix $\mathbf{L}$ denote with $\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$ its eigendecomposition with $\mathbf{U} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N]$ the eigenvectors matrix and $\mathbf{\Lambda}$ the diagonal matrix containing the associated eigenvalues λ_i, $i = 1, \ldots, N$. Then, for undirected graphs the GFT of the graph signal $\boldsymbol{s} : \mathcal{V} \rightarrow \mathbb{R}^N$ is defined as the projection of $\boldsymbol{x}$ on the eigenvectors of $\mathbf{L}$, i.e.

$$\hat{s}(i) =< \boldsymbol{u}_i, \boldsymbol{s} >= \sum_{n=1}^{N} s(n)u_i(n), \quad i = 1, \ldots, N$$

in matrix form

$$\hat{\boldsymbol{s}} = \mathbf{U}^T \boldsymbol{s}. \tag{1}$$

The Inverse GFT is given by

$$\boldsymbol{s} = \mathbf{U}\hat{\boldsymbol{s}}. \tag{2}$$

Note that (2) can be obtained multiplying (1) by $\mathbf{U}$ and using the property $\mathbf{U}\mathbf{U}^T = \mathbf{I}$, so that

$$\mathbf{U}\hat{\boldsymbol{s}} = \mathbf{U}\mathbf{U}^T \boldsymbol{s} = \boldsymbol{s} \tag{3}$$

b) A graph signal $\boldsymbol{s}$ is band-limited over a graph if its GFT $\hat{\boldsymbol{s}}$ is sparse, i.e. $\hat{\boldsymbol{s}}$ has only K non-zero coefficients. Then, if the graph signal is spanned on the first K-eigenvectors of $\boldsymbol{u}_i$, $i = 1, \ldots, K$, the graph signal can be represented as

$$\boldsymbol{s} = \sum_{i=1}^{K} \hat{s}_i \boldsymbol{u}_i.$$