Fourier series and signal spaces

slide set # 3

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Summary of the presentation

In the first part, after an introduction to phasors notation, the representation of periodic signals as an ordered set of Fourier coefficients is given, together with their reconstruction formula known as Fourier series. Then the alternative representations for real signals are given, as well as the effects of using only a limited set of coefficients. This section closes with the proof of the Parseval's theorem which gives a typical effect of the exponentials orthogonality property

The second part of this series of slides deals with the concepts of vector algebra when applied to signal spaces and, with the excuses of motivating the expression of the Fourier series, we take a guided tour through notions as the basis of representation of a vector space, the norm of a vector, inner product spaces and Schwartz inequality

These concepts are then applied to the spaces of periodic, energy and power signals, after having defined for them the formula that evaluates the inner product, and showing how many of the properties valid for signals are correlated in a simple way with the particularities of metric spaces. Furthermore, its shown how any linear transformation or operator can be thought of as the result of evaluating an inner product, so that many signal processing results such as Fourier transform, convolution, signal correlation, fall into this case

Content index

Now we talk about...

Phasor

How to convert a complex number into a cosine

- The complex number $\underline{x} = Ae^{j\varphi}$ is called a *phasor* when used as a shorthand notation to represent amplitude A and phase φ of a cosine wave $x(t)$ at frequency f_0, as given by the expression

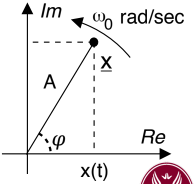
$$x(t) = \Re\{\underline{x} \cdot e^{j2\pi f_0 t}\} = A\cos(2\pi f_0 t + \varphi)$$

  - taking f_0 fixed, A and φ are the information encoding parameters
  - the time-varing $e^{j2\pi f_0 t}$ complex number *could* be called *a rotor* by virtue of the rotation it *imprints* on the constant $\underline{x}$ by which it is multiplied

- The result can be graphically interpreted as having impressed on the $\underline{x}$ phasor a *counterclockwise* rotation with angular velocity

$$\omega_0 = 2\pi f_0 \ \left[{}^{radiants}/_{second}\right]$$

  and having projected the result on the real axis
  - the *counterclockwise* rotation is associated with an angle which *is increasing*



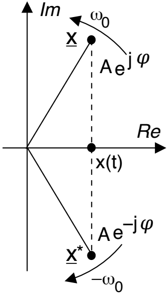
The unexpected entrance of negative frequencies

As $\underline{a} + \underline{a}^* = 2\Re\{\underline{a}\}$ is true, then $\Re\{\underline{a}\} = \frac{1}{2}(\underline{a} + \underline{a}^*)$, and $A\cos(2\pi f_0 t + \varphi)$ can be evaluated from phasor $\underline{x}$ and rotor $e^{j2\pi f_0 t}$ as

$$x(t) = A\cos(2\pi f_0 t + \varphi) = \Re\{\underline{x} \cdot e^{j2\pi f_0 t}\} = \frac{1}{2}\{\underline{x}\, e^{j2\pi f_0 t} + \underline{x}^* e^{-j2\pi f_0 t}\}$$

that is involving a rotor with *negative* angular velocity $-\omega_0$, hence

- the clockwise-rotating $\underline{x}^* e^{-j2\pi f_0 t}$ has the imaginary part with sign always opposite to the first, causing them to *elide*
- the sum of $\underline{x}\, e^{j2\pi f_0 t}$ by it's conjugate gives $2\Re\{\underline{x} e^{j2\pi f_0 t}\}$, hence the above result
  - such a result coincides with the expression of the *Fourier series* expansion for a single frequency signal! (more on this later, page 23)



But why a *negative* frequency??

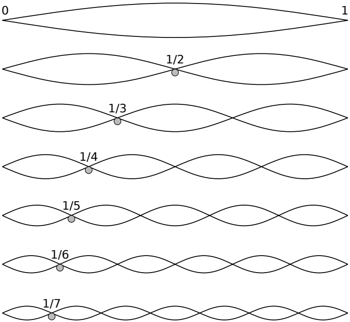
Since time t always *increase*, i.e. runs in the positive direction, the minus sign of $-2\pi f_0 t$ *must* arise due to a frequency which is *negative*

Now we talk about...

The harmonics of a periodic signal

A *periodic* signal $x(t)$ repeatedly assumes the same values after a time interval which is any integer multiple of its *period* T; the inverse of T is its *fundamental frequency* $F = \frac{1}{T}$ or *first harmonic*, expressed in Hertz (or $\sec^{-1}$)

- signal's harmonics are nF frequencies integer multiple of the fundamental F
- think at F as the lowest frequency at which a guitar string oscillates when it is plucked and no fret has been pressed
- any guitarist knows how to produce the n^{th} harmonic sound by touching (not pressing) the string in a position that is at $1/n$ of its length
- harmonic frequencies are related to musical notes as follows (starting from C_2)



harmonic #	1	2	3	4	5	6	7	8	9	10
note	C_2	C_3	G_3	C_4	E_4	G_4	$B^\flat_4$	C_5	D_5	E_5
tone intervals from F	1	8	12	16	18	20	$23^\flat$	24	25	26

The table above is about *intonazione naturale*, but things are more complicated

Fourier series expansion and coefficients

A periodic signal $x(t)$ **can be reconstructed** by the knowledge of an *infinite sequence* (or ordered set) of complex values $\{X_n\}$ called *Fourier coefficients*, and each X_n is obtaines starting from a signal period as

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,\mathrm{e}^{-j2\pi nFt}dt \quad n = 0, \pm 1, \pm 2, \cdots \tag{1}$$

- ...psst... please note that X_n is a *complex number* just because $\int_{-T/2}^{T/2} x(t)\,\mathrm{e}^{-j2\pi nFt}dt = \int_{-T/2}^{T/2} x(t)\cos 2\pi nFt\,dt - j\int_{-T/2}^{T/2} x(t)\sin 2\pi nFt\,dt$

In fact, X_n values allow to *reconstruct* $x(t)$ as a *linear combination* of an infinite number of complex exponential functions $\mathrm{e}^{j2\pi nFt}$. The resulting reconstruction formula is known as the *Fourier series*, written as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n\,\mathrm{e}^{j2\pi nFt} \tag{2}$$

- Note the usage of uppercase X for the frequency description of a lowercase x time signal
- Here is a nice website where to experiment with Fouries series analysis for some waveforms
- A *geometrical* justification for these formulas will be given later (pag. 38)

Some properties of the Fourier series expansion

- the sum of the symmetric terms $X_n e^{j2\pi nFt} + X_{-n} e^{-j2\pi nFt}$ is called an *harmonic component* of $x(t)$ at frequency $f = nF$
  - if $x(t)$ is real we will see that the harmonic components are equal to $2|X_n|\cos(2\pi nFt + \varphi_n)$ where φ_n is the phase of X_n;
- the coefficient $X_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$ is the *mean value* of $x(t)$
- the Fourier coefficients X_n can also be calculated for a signal with *finite duration* T
  - Fourier series (2) will then return a periodic signal
- the Fourier series gives
  - exact values at time instants where $x(t)$ is continuous, and
  - a value equal to the average of values at the extremes in correspondence to discontinuities of the first kind
- the complex coefficients X_n can be expressed in terms of the corresponding real and imaginary parts such as $X_n = \Re\{X_n\} + j\Im\{X_n\}$, or in exponential form $X_n = M_n e^{j\varphi_n}$ in which

$$\begin{cases} M_n = \sqrt{\Re\{X_n\}^2 + \Im\{X_n\}^2} = |X_n| & \textit{is the module spectrum} \\ \varphi_n = \arctan\frac{\Im\{X_n\}}{\Re\{X_n\}} & \textit{is the phase spectrum} \end{cases}$$

Now we talk about...

Conjugate or Hermitian symmetry

For the Fourier coefficients of a real signal

- If the periodic signal $x(t)$ *is real*, its Fourier coefficients satisfy the *conjugate symmetry* property, expressed as
$$X_n = X^*_{-n}$$
i.e. the coefficient with index n is the conjugate of the one with index $-n$, or equivalently $\Re\{X_n\}$ is *even* and $\Im\{X_n\}$ is *odd*. This implies that X_0 *is real*
- In fact, as $e^{-j2\pi nFt} = \cos 2\pi nFt - j \sin 2\pi nFt$, it happens that (2) becomes
$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(2\pi nFt)\,dt - j\frac{1}{T}\int_{-T/2}^{T/2} x(t)\sin(2\pi nFt)\,dt$$
so that (cos is *even*) the real part of X_n doesn't change if n changes sign, while its imaginary part, on the other hand, changes sign (sin is *odd*)

The above implies a similar property also for module and phase of X_n, so that

> *if $x(t)$ is real then the coefficients X_n have even real part and odd imaginary part, and also module even and odd phase*

from which also descends

> *if $x(t)$ is not only real but is also* EVEN *then its coefficients X_n are just real (even), whereas if $x(t)$ is real* ODD, *the X_n are just imaginary (odd)*

Interpretation of Fourier coefficients as phasors

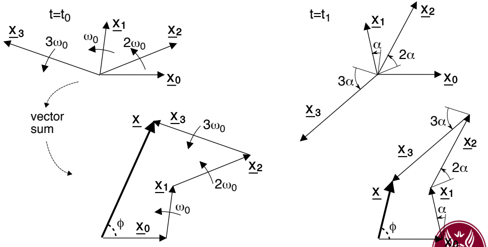
For real signals

The conjugate symmetry property $X_{-n} = X_n^*$ valid for real signals allows to rewrite (apart of the mean X_0) the Fourier series (2) as

$$x(t) = \sum_{n=1}^{\infty} \left(X_n \, \mathrm{e}^{j2\pi nFt} + X_{-n} \, \mathrm{e}^{-j2\pi nFt} \right) = 2\Re\left\{ \sum_{n=1}^{\infty} X_n \, \mathrm{e}^{j2\pi nFt} \right\} \qquad (3)$$

in which (for a given t) every term n of the sum *is real* because it's the sum of two conjugated numbers; in other words, the whole sum is twice the real part of the sum of the terms only with $n > 0$.

Being aware that each phasor of the sum rotates with an angular velocity $\omega_n = 2\pi nF$, the *vector sum* of the first three terms $X_n \, \mathrm{e}^{j2\pi nFt}$, evaluated for two consecutive instants t_1 and $t_2 > t_1$ is drawn, showing how in the interval $\tau = t_2 - t_1$ the phasors $\underline{X}_2$ and $\underline{X}_3$ are rotated by an angle $n\alpha$ which is a multiple of angle $\alpha = 2\pi F\tau$ by which $\underline{X}_1$ has rotated



...if you feel confused, check out these too: [1], [2], [3]

Trigonometric series

For periodic and real signals

- Again by virtue of the conjugate symmetry property of X_n for real signals we can write $X_{\pm n} = M_n e^{\pm j\varphi_n}$ so that terms in round brackets at eq. (3) become

  $$M_n \left(e^{j(2\pi nFt+\varphi_n)} + e^{-j(2\pi nFt+\varphi_n)}\right) = M_n 2\cos\left(2\pi nFt + \varphi_n\right)$$

  hence the Fourier series (2) can be written in terms of a sum of cosines, each with different amplitude and phase

  $$x\left(t\right) = X_0 + 2\sum_{n=1}^{\infty} M_n \cos\left(2\pi nFt + \varphi_n\right)$$

- If we now otherwise start from the other way of expressing the conjugate symmetry property, that is $X_{\pm n} = R_n \pm jI_n$, we can arrive at

  $$x\left(t\right) = X_0 + \sum_{n=1}^{\infty} \left\{2R_n \cos\left(2\pi nFt\right) - 2I_n \sin\left(2\pi nFt\right)\right\}$$

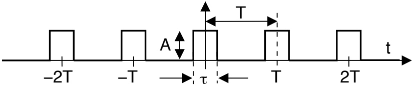
  where the signal is expressed in terms of both cosines and sines
  - note that a weighted sum of cos and sin *with the same frequency* creates a new sinusoidal waveform (with the same frequency) and whose phase depends on the weights - for instance, look here
- So the Fourier series for real signals can be traced back to an infine sum of trigonometric functions
  - in particular if the signal is even it will be a series of cosines (with zero phase), or if the signal is odd, a series of only sines (again with a null phase)

Fourier series of a rectangular wave

Let's perform the analysis of a real periodic signal made by repeating with period T a rectangular impulse of duration $\tau < T$, written as

$$x(t) = A\sum_{n=-\infty}^{\infty}\mathrm{rect}_\tau(t - nT)$$

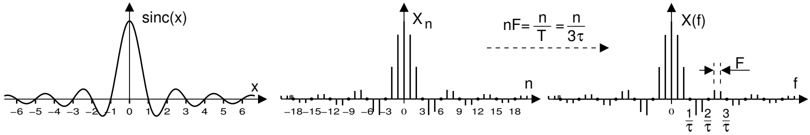
and depicted on the right for $T = 3\tau$



The computation of its Fourier coefficients ends with the expression

$$X_n = A\frac{\tau}{T}\mathrm{sinc}(nF\tau) \tag{4}$$

in which the *cardinal sine* function $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ is sampled with a regularly spaced argument $nF\tau = n\frac{\tau}{T} = n\frac{\tau}{3\tau} = \frac{n}{3}$ as shown below



- X_n gives the contribution at frequency $nF = \frac{n}{T}$, X_0 equals the mean value $\frac{A}{3}$
- harmonics are spaced by $F = \frac{1}{T}$
- frequencies equal to an integer multiple of $\frac{1}{\tau}$ have the respective $X_n = 0$

Fourier series of a rectangular wave - 2

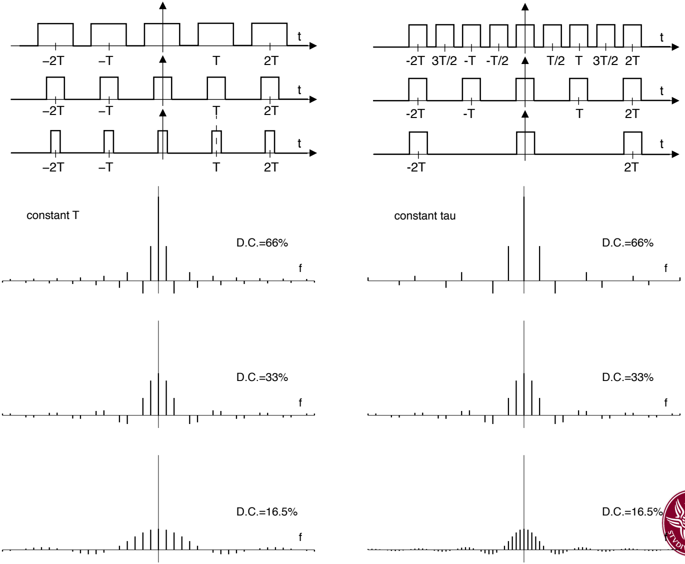
Let's play a little

Left - fixed T with decreasing τ

Right - fixed τ with increasing T

Left - sinc envelope widens, fundamental and harmonic freqs stay the same

Right - sinc envelope stay the same, fundamental freq. reduces and harmonics get nearer



Now we talk about...

Truncated Fourier series

Since infinity is a mathematical abstraction, the infinite Fourier series cannot be calculated in any way. So let's see what happens if the summation is instead limited only to coefficients X_n with $-N \le n \le N$, working for example on a zero-mean *square* wave

$$x(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}_\tau(t - kT) - 0.5$$

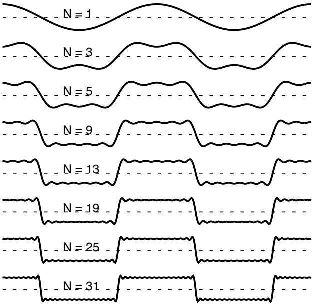
with $\tau = T/2$, for which (4) becomes $X_n = \frac{1}{2}\mathrm{sinc}\left(\frac{n}{2}\right)$, non-zero only with *n odd*, or

$$X_n = \begin{cases} \frac{1}{2}\frac{\sin \pi\frac{n}{2}}{\pi\frac{n}{2}} = \frac{(-1)^{\frac{n-1}{2}}}{\pi n} & \textit{with} \quad \mathrm{n}\ \textit{odd} \\ \textit{zero} & \textit{with} \quad \mathrm{n}\ \textit{even} \end{cases}$$

and as $x(t)$ is real even, we can approximate it as a truncated series of cosines

$$\widehat{x}_N(t) = \sum_{n=1, n\ odd}^{N} 2X_n \cos(2\pi n F t)$$

producing the result shown for different choices of N



- With the increase of N the reconstruction is more and more accurate, except for oscillations near the discontinuity, which take the name of Gibbs phenomenon
- Such a phenomenon doesn't occurr for smoother waveforms such as a triangular one, as can be experienced through the Falstad's on-line calculator

Now we talk about...

Parseval's theorem

- We have already defined the power $\mathcal{P}_x$ of a periodic signal $x(t)$ as the mean integral sum of its squared values, or $\mathcal{P}_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt$
- Parseval's theorem makes it equal to an infine but discrete sum taken on the squares of its Fourier coefficients X_n, i.e.

$$\mathcal{P}_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |X_n|^2 \tag{5}$$

- To prove this to be true, let's elaborate the steps for the more general case of a complex signal:

$$\begin{aligned} \mathcal{P}_x &= \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt = \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \left[\sum_n X_n \, \mathrm{e}^{j2\pi nFt}\right] \left[\sum_m X_m^* \, \mathrm{e}^{-j2\pi mFt}\right] dt = \\ &= \sum_n \sum_m X_n X_m^* \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{e}^{j2\pi(n-m)Ft} dt = \\ &= \sum_n \sum_m X_n X_m^* \delta_{n,m} = \sum_{n=-\infty}^{\infty} X_n X_n^* = \sum_{n=-\infty}^{\infty} |X_n|^2 \end{aligned} \tag{6}$$

Surprised? Let's see what has happened...

Orthogonality of complex exponentials

- The simplification of the double sum in (6) into one is due to the result

$$\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{e}^{j2\pi(n-m)Ft}dt = \delta_{n,m} = \begin{cases} 0 & with \quad m \neq n \\ 1 & with \quad m = n \end{cases} \tag{7}$$

  forcing to zero all terms of the double sum with $m \neq n$
- in fact, by setting $k = n - m$ the integrating function $\mathrm{e}^{j2\pi kFt}$ becomes $\cos\left(2\pi\frac{k}{T}t\right) + j\sin\left(2\pi\frac{k}{T}t\right)$, and therefore equal to 1 when $k = 0$, and (co)sinusoidal with period $\frac{T}{k}$ for $k \neq 0$. Therefore
  - for $m = n$ ($k = 0$) the integral evaluates as T (or one if divided by T)
  - when instead $k \neq 0$, an integer number of k periods of the integrand falls within the integration interval T, giving a null mean value: this involves the disappearance of the terms with $m \neq n$ from the double sum
- Based on the principles of vector algebra illustrated on page 32, the property (7) is indicated as *the orthogonality of complex exponentials*
  - or, even better, of complex exponentials which are *different harmonics of a common fundamental*

Power spectrum of periodic signals

Any $|X_n|^2$ term in the $\mathcal{P}_x = \sum_{n=-\infty}^{\infty} |X_n|^2$ Parseval's sum is the power $\mathcal{P}_n$ of the $X_n\, \mathrm{e}^{j2\pi nFt}$ term in the $x(t) = \sum_n X_n\, \mathrm{e}^{j2\pi nFt}$ Fourier series for $x(t)$, i.e.

$$\mathcal{P}_n = \frac{1}{T}\int_{-T/2}^{T/2} \left[X_n\, \mathrm{e}^{j2\pi nFt}\right]\left[X_n^*\, \mathrm{e}^{-j2\pi nFt}\right] dt = \frac{|X_n|^2}{T}\int_{-T/2}^{T/2} dt = |X_n|^2$$

and therefore

> *The total power $\mathcal{P}_x$ of a periodic signal $x(t)$ is equal to the sum of the powers of its component harmonics*

This is a consequence of the orthogonality of complex exponentials, since the square of a sum is generally *not* equal to the sum of the squares

- say, $(a+b)^2 = a^2 + b^2 + 2ab \neq a^2 + b^2$; equality occurs only if the addends are *orthogonal*

The sequence $\{\mathcal{P}_n\} = \left\{\ldots, |X_{-k}|^2, \ldots, |X_0|^2, \ldots, |X_k|^2, \ldots\right\}$ therefore represents how the total power *is distributed* between the different harmonics at frequency $f = nF$, and is called the *power spectrum* of signal $x(t)$

- we observe that the terms $\mathcal{P}_n = |X_n|^2$ are necessarily real and positive;
- furthermore, if $x(t)$ is real then the conjugate symmetry property implies that $|X_n|^2 = |X_{-n}^*|^2 = |X_{-n}|^2$, so $\mathcal{P}_n = \mathcal{P}_{-n}$; therefore
- a *real* signal is characterized by an *even* power spectrum

Recap: Fourier series and power spectra of a cosine

Let's evaluate the Fourier coefficients and the power spectra of a sinusoidal waveform expressed in the more general form as $x(t) = A\cos(2\pi Ft + \varphi)$

- computation of $X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, \mathrm{e}^{-j2\pi nFt} dt$ with $T = \frac{1}{F}$ gives
$$X_n = \frac{A}{2T}\int_{-T/2}^{T/2} \left(\mathrm{e}^{j2\pi Ft}\mathrm{e}^{j\varphi} + \mathrm{e}^{-j2\pi Ft}\mathrm{e}^{-j\varphi}\right)\, \mathrm{e}^{-j2\pi nFt} dt$$
which is zero for $|n| > 1$ due to the orthogonality of harmonic exponentials, and zero for $n = 0$ because $\cos(2\pi Ft)$ and $\sin(.)$ have zero mean over an entire period
- the integral above splits in the sum of two, each evaluated in the next two rows:
  - with $n = 1$ we have $\mathrm{e}^{j\varphi}\int_{-T/2}^{T/2} \mathrm{e}^{j2\pi Ft}\, \mathrm{e}^{-j2\pi Ft} dt = \mathrm{e}^{j\varphi}\int_{-T/2}^{T/2} 1 \cdot dt = \mathrm{e}^{j\varphi}\cdot T$ and
  $\mathrm{e}^{-j\varphi}\int_{-T/2}^{T/2} \mathrm{e}^{-j2\pi Ft}\, \mathrm{e}^{-j2\pi Ft} dt = \mathrm{e}^{-j\varphi}\int_{-T/2}^{T/2} \mathrm{e}^{-j2\pi 2Ft} dt = 0$ since
  $\mathrm{e}^{-j2\pi 2Ft} = \cos 2\pi 2Ft + j\sin 2\pi 2Ft$ has zero mean over the interval $T = \frac{1}{F}$, so that
  $$X_1 = \frac{A}{2T} T\mathrm{e}^{j\varphi} = \frac{A}{2}\mathrm{e}^{j\varphi}$$
  - with $n = -1$ we can repeat computations, or remember that for real signals $X_{-1} = X_1^*$ and therefore $X_{-1} = \frac{A}{2}\mathrm{e}^{-j\varphi}$
- the power spectra is then evaluated as
$$P_1 = |X_1|^2 = \left|\frac{A}{2}\mathrm{e}^{j\varphi}\right|^2 = \frac{A^2}{4}; \quad P_{-1} = P_1 \text{ (as for a real signal } x(t))$$
and the total power of a cosine is
$$P_x = |X_1|^2 + |X_{-1}|^2 = 2\frac{A^2}{4} = \frac{A^2}{2}$$

Now we talk about...

The signals space, a unifying synthesis

- It's a way to approach the analysis of signals by representing them as elements of a *vector space*, thus extending to the former the algebraic and geometric properties which are valid for the latter
- First we will briefly revisit the theory of algebraic spaces for which a *scalar (or dot) product* operator is defined, and then we'll identify the correspondences between the properties of vectors and those of signals
- In this way a unifying synthesis is obtained in relation to the basic concepts of *representation basis*, *orthogonality*, and *unitary transformation*
- Finally, we will mention how many of the integrals studied in the text (*Fourier coefficients and transform*, *cross-energy, convolution*, *correlation*) can be traced back to the computation of a scalar product

Now we talk about...

Linear and vector spaces

A set $\mathcal{A}$ of elements is called *linear space* (or *vector space*) on a field K (K can be the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers) when

- *the sum* of elements $\overline{x}, \overline{y} \epsilon \mathcal{A}$ and their *multiplication* by a coefficient $\lambda \epsilon K$ are defined, the result of which is yet another element of the set
- giving the space $\mathcal{A}$ the properties of an *algebraic structure*

In addition to the *Euclidean space* $\mathbb{R}^n$ from which this notion originates, signals belonging to the same class are also elements of a *common vector space*, as in the case of periodic signals with period T, or of energy signals, or power

Basis of representation

A vector space is of dimension n if, choosing any subset $\mathcal{B} \subseteq \mathcal{A}$ of n linearly independent vectors $\overline{u}_i$ (ie none of which is a linear combination of the others), any other vector $\overline{x} \in \mathcal{A}$ can be espressed as a linear combination

$$\overline{x} = \sum_{i=1}^{n} x_i \overline{u}_i$$

with coefficients $x_i \in K$

- the vectors $\overline{u}_i \in \mathcal{B}$ are called the *representation basis* for the space $\{\mathcal{A}, K\}$, and the coefficients $\{x_1, x_2, \cdots, x_n\}$ are the $\overline{x}$ vector coordinates on such basis

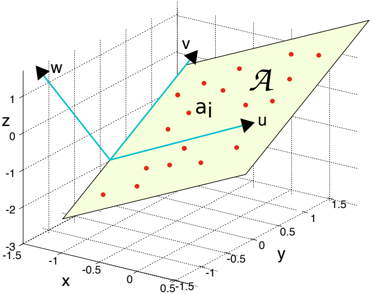
Dimensions of a vector space

and linear independence of the basis vectors

In this image the set $\mathcal{A}$ is made up of the points a_i lying on the green plane

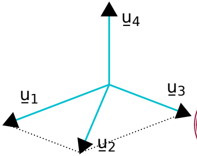
Although each point a_i is described by *a triple* (x_i, y_i, z_i) in the reference system (x, y, z), in reality space $\mathcal{A}$ is only **two-dimensional**, and points can be referenced by just *a pair* of new coordinates (u_i, v_i) in the different reference system (u, v, w)

A way to obtain (u_i, v_i) from (x_i, y_i, z_i) is called *a linear transformation*, which in R^3 is a multiplication by matrix, plus a translation



To be **linearly independent**, the vectors $\overline{u}_i$ of a basis must be such that $\sum_{i=1}^{n} x_i \overline{u}_i = 0$ is satisfied *only when all the x_i are zero*

Example: $\overline{u}_1$, $\overline{u}_2$ and $\overline{u}_3$ are coplanar in $\mathbb{R}^3$ and, therefore, they do not form a basis; conversely, $\overline{u}_4$ and any two of $\{\overline{u}_1, \overline{u}_2, \overline{u}_3\}$ are linearly independent, even if the latter are not orthogonal



But when the linearly independent vectors also are *mutually orthogonal* things go fine (pag. 33)

Normed space

It's a vector space for which a method (*of any kind!*) for evaluating the norm (or length) $\|\overline{x}\|$ of its elements $\overline{x}$ is defined, with properties

1. $0 \leq \|\overline{x}\| < \infty$, with $\|\overline{x}\| = 0$ if and only if $\overline{x} = 0$
2. $\|\lambda\overline{x}\| = \lambda\,\|\overline{x}\|$ *homogeneity with* $\lambda \epsilon K$
3. $\|\overline{x} + \overline{y}\| \leq \|\overline{x}\| + \|\overline{y}\|$ *triangle inequality*

by which we obtain a distance measure $d\,(\overline{x}, \overline{y})$ between pairs of vectors as $d\,(\overline{x}, \overline{y}) \doteq \|\overline{x} - \overline{y}\|$. In this way the normed space becomes not only *algebraic* (because it is linear) but also *metric*, with a metric that is *induced* by the norm

Norm for Euclidean spaces

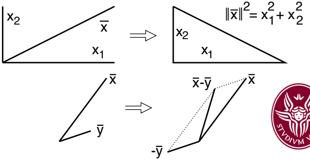
A vector space with (finite) dimensions n is *isomorphic* to $\mathbb{R}^n$, where the norm *of order* 2 for a vector $\overline{x}$ is defined as

$$\|\overline{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

which gives the *Pythagorean theorem*, and induces an *Euclidean distance* as

$$d_2\,(\overline{x}, \overline{y}) = \|\overline{x} - \overline{y}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

Euclidean geometry calculates the distance between two points through the graphic construction of the vectors whose components are the coordinates of the points as in the figure, and therefore $\overline{z} = \overline{x} - \overline{y}$ has components $z_1 = x_1 - y_1$ and $z_2 = x_2 - y_2$.



Now we talk about...

Inner (or dot, or scalar) product

Let us now broaden our point of view and refer to a *generic* linear space $\mathcal{S}$ defined on a real $\mathbb{R}$ or complex $\mathbb{C}$ field K, for which no norm is defined. Instead, we assume that an operator called *scalar product*, or *dot* product, or inner product

$$\langle \overline{x}, \overline{y} \rangle = \lambda$$

is defined, however it is evaluated, as its task is just to associate a pair of vectors $\overline{x}, \overline{y} \in \mathcal{S}$ to a scalar $\lambda \in K$, satisfying the properties

1. $\langle \overline{x}, \overline{x} \rangle \geq 0$ is real, with $\langle \overline{x}, \overline{x} \rangle = 0$ if and only if $\overline{x} = 0$
2. $\langle \overline{x}, \overline{y} \rangle = \langle \overline{y}, \overline{x} \rangle^*$ *conjugate symmetry*, or *commutative* property if $K = \mathbb{R}$
3. $\langle a\overline{x} + b\overline{y}, \overline{z} \rangle = a \langle \overline{x}, \overline{z} \rangle + b \langle \overline{y}, \overline{z} \rangle$ *linear* and *distributive* property
    1. $\langle \overline{z}, a\overline{x} + b\overline{y} \rangle = a^* \langle \overline{z}, \overline{x} \rangle + b^* \langle \overline{z}, \overline{y} \rangle$ *conjugated linear* in the second argument (from ② and ③, if $K = \mathbb{C}$)

The relation $\langle \overline{x}, \overline{x} \rangle \geq 0$ suggests using the inner product as a measure for the *norm* of a vector $\overline{x}$ by setting $\|\overline{x}\| = \sqrt{\langle \overline{x}, \overline{x} \rangle}$; expecially since the property ③ also corresponds the homogeneity property ② of the norm (page 29)

But in order for the norm induced by the dot product to satisfy the triangular inequality as well, we have to use the results from the next slide, in which the notion of *angle* θ between vectors is also introduced

Schwartz inequality and angles

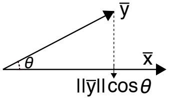
For a scalar product *it can be shown* (in the book) that

$$|\langle \overline{x}, \overline{y}\rangle|^2 \leq \langle \overline{x}, \overline{x}\rangle \cdot \langle \overline{y}, \overline{y}\rangle \quad \text{or} \quad |\langle \overline{x}, \overline{y}\rangle| \leq \|\overline{x}\| \cdot \|\overline{y}\|$$

known as *Schwartz inequality*: the inner product between vectors is never greater than the product of their respective lengths, with equal sign only if the vectors are proportional, i.e. if $\overline{x} = \alpha\overline{y}$, or in geometric terms, they are *parallel*

Angle between vectors

- from the above we get $0 \leq \frac{|\langle \overline{x}, \overline{y}\rangle|}{\|\overline{x}\| \cdot \|\overline{y}\|} \leq 1$, and the extremes 0 and 1 express the conditions of *orthogonality* and *parallelism* between $\overline{x}$ and $\overline{y}$
- only one θ angle is such that $\cos\theta = \frac{|\langle \overline{x}, \overline{y}\rangle|}{\|\overline{x}\| \cdot \|\overline{y}\|}$, therefore θ is considered as the angle between $\overline{x}$ and $\overline{y}$, and so



  or
  $$|\langle \overline{x}, \overline{y}\rangle| = \|\overline{x}\| \cdot \|\overline{y}\| \cdot \cos\theta$$

  the scalar product is equal to the product of the modules times the cosine of the angle between the vectors
- by virtue of the last result, it follows that

  Two vectors are said to be orthogonal if their scalar product is zero, or
  $$\langle \overline{x}, \overline{y}\rangle = 0$$

Scalar product between vectors

expressed on the same orthogonal basis

Evaluation of vector coefficients by scalar product

- As we know the vectors $\overline{x}$ of a linear $n-$dimensional space $\mathcal{A}$ can be expressed by their coefficients $\{x_1, x_2, \cdots, x_n\}$, letting us to write
$$\overline{x} = \sum_{i=1}^{n} x_i \overline{u}_i$$
where $\overline{u}_i \in \mathcal{B}$, and $\mathcal{B} \subset \mathcal{A}$ is a representation basis of $\mathcal{A}$
- the coefficients x_i can be obtained, if the other condition $\langle \overline{u}_i, \overline{u}_j \rangle = 0$ for all $i \neq j$ holds (making the basis *orthogonal*), by evaluation of the inner product (or projection) between $\overline{x}$ and $\overline{u}_i$, i.e
$$x_i = \frac{\langle \overline{x}, \overline{u}_i \rangle}{\|\overline{u}_i\|^2}$$

Evaluation of the scalar product by vector coefficients

- The *scalar product* between vectors $\overline{x} = \sum_{i=1}^{n} x_i \overline{u}_i$ and $\overline{y} = \sum_{i=1}^{n} y_i \overline{u}_i$ with coefficients $\{x_i\}$ and $\{y_i\}$ can be evaluated in terms of a *scaled* scalar product between the coefficients of the vectors, that is:
$$\langle \overline{x}, \overline{y} \rangle = \sum_{i=1}^{n} x_i y_i^* \|\overline{u}_i\|^2$$
for a proof, write $\langle \overline{x}, \overline{y} \rangle = \left\langle \sum_{i=1}^{n} x_i \overline{u}_i, \sum_{j=1}^{n} y_j \overline{u}_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j^* \langle \overline{u}_i, \overline{u}_j \rangle$
and then cancel out the terms $i \neq j$ because of orthogonality

Calculation of the norm with an orthonormal basis
and induced distance measure

If the vectors $\overline{u}_i$, besides being orthogonal they also have a norm equal to one, i.e. $\|\overline{u}_i\|^2 = \langle \overline{u}_i, \overline{u}_i \rangle = 1$, then they are called *unitary*, the base is said to be *orthonormal*, and the previous expression for the inner product between $\overline{x}$ and $\overline{y}$ becomes

$$\langle \overline{x}, \overline{y} \rangle = \sum_{i=1}^{n} x_i y_i^*$$

while for the *quadratic* norm of a generic vector $\overline{x}$ we get

$$\|\overline{x}\|^2 = \langle \overline{x}, \overline{x} \rangle = \sum_{i=1}^{n} x_i x_i^* = \sum_{i=1}^{n} |x_i|^2$$

which corresponds to the square of the *Euclidean norm*

Distance measure induced by the scalar product

It is expressed as the *norm of the difference vector*, or

$$d(\overline{x}, \overline{y}) = \sqrt{\|\overline{x} - \overline{y}\|^2} = \sqrt{\langle \overline{x} - \overline{y}, \overline{x} - \overline{y} \rangle} = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

which in turn corresponds to the distance of order $p = 2$ for Euclidean spaces. In the case of vectors with complex coefficients we have

Application: the matrix of a change of coordinates

Let us consider a vector $\overline{x} = [x_1 x_2 \cdots x_n]^T$, $\overline{x} \in \mathbb{R}^n$, whose coefficients refer to a coordinate system U defined by an arthogonal representation basis $\{\overline{u}_1 \overline{u}_2 \cdots \overline{u}_n\}$

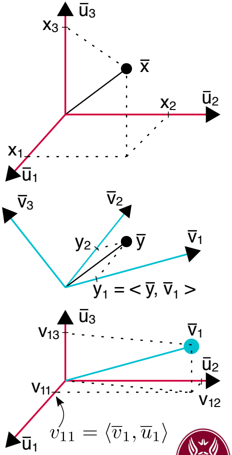
Linear algebra theory says that the coefficients of $\overline{x}$ in a new coordinate system V with base $\{\overline{v}_1 \overline{v}_2 \cdots \overline{v}_n\}$, which shares the same origin as U, are given by a matrix product, that is

$$\overline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = A\overline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

But: what values should the a_{ij} assume?

Recall that the coefficient $y_i = \langle \overline{y}, \overline{v}_i \rangle$ is the dot product of $\overline{y}$ and $\overline{v}_i$, which can be evaluated as $\sum_{j=1}^{n} x_j v_{ij}^*$, i.e. using the representation coefficients of $\overline{y}$ and $\overline{v}_i$ in the common original orthogonal base $U = \{\overline{u}_1 \overline{u}_2 \cdots \overline{u}_n\}$

Thus, the i−th row of A is made up of the coefficients of $\overline{v}_i$ expressed with respect to the vectors $\overline{u}_i$ of the previous base, i.e. $a_{ij} = v_{ij} = \langle \overline{v}_i, \overline{u}_j \rangle$



Now we talk about...

From vector to signal spaces

The results given for spaces with a finite number n of dimensions remain valid even when $n \to \infty$, as for the spaces of periodic, energy or power signals, such as

- a space with infinite (but *countable*) dimensions is the one whose coefficients set $\{X_i\}$ (with $i = 1, 2, \cdots, \infty$) for vectors $\overline{X}$ is formed by the Fourier coefficients of a periodic signal $x(t)$
  - such that the squared norm $\left\|\overline{X}\right\|^2 = \sum_{i=1}^{\infty} |X_i|^2$ converges to a finite number, the signal power $\mathcal{P}_x$, as given by the Parseval's theorem
- another space with infinite (and *uncountable*) dimensions is composed of signals $\overline{x} = x(t)$ *for* $\forall t$ whose square is summable, that is, energy signals
  - giving to its squared norm $\|x(t)\|^2 = \int x^2(t)\, dt$ the meaning of energy $\mathcal{E}_x$, and leading to the definition of squared distance $d^2(x(t), y(t))$ between signals as the energy of the difference signal $\int |x(t) - y(t)|^2\, dt$

Although we will not go into the subtle analytical details, the type of equivalences just mentioned are a consequence of adapting the definition of inner product to the infinite dimensions case, allowing to associate a unifying geometric interpretation to the obtainable relations

Space of periodic signals with period T

When this space is equipped with an inner product between $x(t)$ and $y(t)$ in the form

$$\langle x(t), y(t)\rangle = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt$$

we obtain that

- when calculated for $y(t) = x(t)$ it gives $\mathcal{P}_x$ as the $x(t)$ *quadratic norm*

  $$\mathcal{P}_x = \|x(t)\|^2 = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt$$

  - the norm of $x(t) = \|x(t)\| = \sqrt{\mathcal{P}_x}$ is called the *RMS* or *effective* value of $x(t)$, i.e. the value of a *constant* signal which has the same norm *(power)*
- the orthonormal basis of this space is made by $u_n = \mathrm{e}^{j2\pi nFt}$ signals
  - in fact it results $\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{e}^{j2\pi(n-m)Ft} dt = \delta_{n,m}$ (Kronencker delta)
  - the Fourier coefficients evaluation formula $X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, \mathrm{e}^{-j2\pi nFt} dt = \langle \overline{x}, \overline{u}_n \rangle$ is the dot product in between the signal vector $x(t)$ and the vectors of the basis
  - the Fourier series $x(t) = \sum_{n=-\infty}^{\infty} X_n\, \mathrm{e}^{j2\pi nFt}$ (eq. (2)) represents the signal in terms of its coefficients with respect of an orthonormal basis
- the Parseval's theorem $\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |X_n|^2$ is nothing else than the calculation of a quadratic norm starting from the coefficients evaluated with respect to an orthogonal basis

Inner product for (energy and power) signals

Even in these two cases it is possible to define an inner product operator

- for energy signals it takes the aspect of

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

  which gives rise to a *quadratic norm* $\|x(t)\|^2 = \langle x(t), x(t)\rangle$ with a value equal to the *signal energy*

$$\mathcal{E}_x = \|x(t)\|^2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

- for *power* signals, on the other hand, the scalar product is written as

$$\langle x(t), y(t)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt$$

  to which corresponds a *quadratic norm* $\|x(t)\|^2 = \langle x(t), x(t)\rangle$ equal to the *signal power*

$$\mathcal{P}_x = \|x(t)\|^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

Schwartz inequality for energy and power signals

- For energy signals, once the definition of inequality $\left|\langle\overline{x}, \overline{y}\rangle\right|^2 \leq \langle\overline{x}, \overline{x}\rangle \cdot \langle\overline{y}, \overline{y}\rangle$ is applied to the expression of the dot product, we get

$$\left|\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt\right|^2 \leq \int_{-\infty}^{\infty} |x(t)|^2\, dt \cdot \int_{-\infty}^{\infty} |y(t)|^2\, dt$$

  i.e. the square of the scalar product between signals (that will be called *cross energy*) is always lesser than the product of their energies, except when one is proportional to the other, ie $x(t) = \alpha y(t)$, and it becomes an equality

- In the power signals case the same words apply, after having replaced the dot product definition, obtaining

$$\left|\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt\right|^2 \leq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt \cdot \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |y(t)|^2\, dt$$

These concepts will find application with regard to *cross energy*, in defining the *correlation* between signals, for *matched filter* analysis ...

But let's jump first into one further speculation, of which we leave out the theoretical rigor: a big part of the relations dealt with in the text *are a form of inner product!*

- which may be why you may want to come back here after studying these

Now we talk about...

Systems, transformations, and functional operators

A *system* or transformation $\mathcal{T}[.]$

- outputs a signal $y(t)$ as a function of an input $x(t)$, or $y(t) = \mathcal{T}[x(t)]$
- i.e. it is a *function of function*, also named a linear *operator* or *functional*
- it maps vectors $\bar{x} \in \mathcal{X}$ (the input signals space) to elements $\bar{y} \in \mathcal{Y}$ (space of output signals)

If $\mathcal{X}$ is *complete* and with an inner product defined, then any functional $\mathcal{T}_\varphi[\bar{x}]$ can be expressed in the form of an *inner product*

$$\mathcal{T}_\varphi[\bar{x}] = \langle \bar{x}, \bar{\varphi} \rangle \tag{8}$$

between input $\bar{x} \in \mathcal{X}$ and a vector-signal $\bar{\varphi}$ which characterizes the transformation

- $\bar{\varphi}$ is sometimes called the *kernel function*, the *integral kernel*, or the *nucleus* of an integral transform (but not in this text)
- If also $\bar{\varphi} \in \mathcal{X}$, eq. (8) can give some well-known results:
  - if $\bar{\varphi} = \bar{x}$ then $\mathcal{T}_{\bar{x}}[\bar{x}] = \langle \bar{x}, \bar{x} \rangle$ is the scalar product of signals with itself, letting you to obtain the *energy* or the *power* of signal $\bar{x}$
  - if $\bar{\varphi} \neq \bar{x}$ but belongs to the same class $\mathcal{X}$ as the input, we get the *cross energy* and *intercorrelation* formulas

Fourier transform as a functional operator

Take now for example the case of the *Fourier transform* that calculates

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt = \left\langle x(t), e^{j2\pi ft} \right\rangle$$

or, in other words, the *projection* of $x(t)$ towards the signal $e^{j2\pi ft}$, which in turn represents the vector $\varphi(t, f)$

It turns out that the complex exponentials $e^{j2\pi ft}$ are an *orthonormal* basis for energy signals, condition that in this space is defined as

$$\langle \varphi(t, f), \varphi(t, \lambda) \rangle = \int_{-\infty}^{\infty} e^{j2\pi ft} e^{-j2\pi \lambda t} dt = \delta(f - \lambda)$$

where $\delta(\cdot)$ is a *Dirac impulse*. From this property of $e^{j2\pi ft}$ it derives both the existence of the Fourier *antitransform*, and the *unitarity* property of the transformation, i.e. of not altering the length of the vectors in passing from the elements of $\mathcal{X}$ (function of t) and those of the image space $\mathcal{Y}$ (function of f), and which gives rise to Parseval's theorem

Delay dependent kernels and the space of operators

In some cases the vector $\bar{\varphi}$ that characterizes the functional (8) depends on a *difference* of variables, that is

$$\varphi(t, \tau) = \varphi(t - \tau)$$

as in the *sieving* operator, or *Hilbert's transform*, or the *convolution* integral: in the latter case $\varphi(t, \tau) = h(t - \tau)$ is directly related to the *impulse response* $h(t)$ that completely characterizes the *system*

If we still are not satified to have verified that a convolution and therefore *a filter* equals a scalar product, and therefore is a functional, or *a system*, then let's broaden the discussion by adding the fact that the set of functionals $\mathcal{T}_\varphi$ that operate on the same *input* space $\mathcal{X}$ constitutes *themseves* an Hilbert space, called the *dual space* $\mathcal{X}^*$ of $\mathcal{X}$, in which the norm is defined as

$$\|\mathcal{T}_\varphi\| = \|\bar{\varphi}\| = \sqrt{\langle \bar{\varphi}, \bar{\varphi} \rangle}$$

and if $\|\bar{\varphi}\| < \infty$ the functional is continuous

Output signals in terms of its basis transform

If $\mathcal{X}$ has an *orthonormal* basis $\{u_i(t)\}$ by which to represent its vectors as $x(t) = \sum_{i=1}^{n} x_i u_i(t)$ where $x_i = \langle x(t), u_i(t) \rangle$, then the functionals of $\mathcal{X}^*$ can *in turn* be represented as a linear combination of signal-vectors

A direct way of proceeding expresses the *output signal* $y(\theta) = \mathcal{T}_{\varphi_{t,\theta}}[x(t)]$ as

$$y(\theta) = \sum_{i=1}^{n} x_i \mathcal{T}_{\varphi_{t,\theta}}[u_i(t)] = \sum_{i=1}^{n} x_i v_i(\theta)$$

where $v_i(\theta)$ is the result of the functional $\mathcal{T}_\varphi$ applied to the vectors of the base $u_i(t)$ of $\mathcal{X}$, which therefore operates by adding its *response vectors* $v_i(\theta)$ (which belong to $\mathcal{Y}$) with weights equal to the components x_i of $x(t)$ *projected* on the $u_i(t)$

Example The convolution integral $y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau$ expresses the output $y(t)$ as a linear combination of the $h(t-\tau)$ effect of impulses $\delta(t)$, which constitute an orthonormal basis for $x(t)$ in terms expressed by the sieving property $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

However, it is not certain that $\{v_i(\theta)\}$ form a basis of linearly independent vectors fo $\mathcal{Y}$, but... well, this is really far beyond of the intentions of this exposition!